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The reconstruction of controls by regularization methods with uneven stabilizers☆

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ABSTRACT

The problem of reconstructing a previously unknown control (parameter) of a dynamical system using the results of approximate observations of the motion of this system is considered. It is proposed to use static and dynamic methods to solve this problem which, in their implementation, utilize the method of Tikhonov regularization with a stabilizer containing a variation of the simulating subsidiary control (parameter).The use of such a non-differentiable stabilizer enables one to obtain more refined results than the approximation of the required control in Lebesgue spaces. In particular, the piecewise-uniform convergence of the regularized approximations can be successfully substantiated by this route, which opens up the possibility of numerically reconstructing the fine structure of the required control.

The problem of the regeneration (reconstruction or identification) of previously unknown controls (parameters) operating in a controlled dynamical system is considered. The control actions in the dynamical system can be unknown in advance and must be determined from the results of observations of the object, in particular, from the results of approximate measurements of the current phase positions of the system. The reconstructed controls can then be used to estimate the characteristics of the controlled object, the operational acceptance of the solutions or more adequate modelling. It is well known that the problem considered is ill-posed and its solution requires the use of regularization methods.[1–3] Reconstruction problems of this kind for dynamical systems have been studied in different formulations in control theory, the theory of differential games, and estimation and identification theory.[4–13]

Two methods, a static method and a dynamic method, are proposed for solving the problem. It is proposed to use the Tikhonov variational method to solve the problem by the static method. The essence of the Tikhonov method is the minimization of a certain suitable discrepancy functional in the set of permissible controls. From the point of view of control theory, this method can be classified as a static reconstruction method. In solving the reconstruction problem by this method, the results of approximate measurements of the current phase positions of the system, accumulated from the observation of the dynamical system during some specified time interval, serve as the initial information for solving it. Here, the reconstruction is accomplished *a posteriori* after the elapse of the corresponding time interval for the observation of the system, using the totality of the information available. A distinctive feature of the static approach to the problem lies in the fact that the data for calculating the controls are known in advance and the reconstruction algorithm does not take account of a possible change in these data during the computational process, and the computational process is not, generally speaking, a single process and can be repeated when necessary. The concepts and methods of preset control theory and the theory of ill-posed problems[1–13] are used to solve the problem. The results of instantaneous approximate measurements of the current phase positions of the system, which are received by the observer of the dynamics during some specified time interval, serve as the initial information for the solution when the reconstruction problem is solved by the dynamic method. Here, the measurements and reconstruction are performed dynamically throughout the course of the process using the instantaneously arriving information. The special feature of the dynamic approach lies in the fact that the data for the calculations can only be admitted during the course of the process and can depend at the present time on how the reconstruction was carried out in the past. The development of this approach is associated with the fact that the need to perform a reconstruction synchronously with the development of a process frequently arises in certain engineering and scientific studies. Similar problems are involved in the mechanics of controlled flight and in operational information retrieval during the creation of technological and manufacturing processes. The concepts and the methods of positional control theory and the theory of ill-posed problems[1–13] are used to solve the reconstruction problem by

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the dynamic method. In this case, regularization is performed locally in a short time interval. Constructive stable regularizing algorithms are constructed for solving a problem by one or the other of the methods. Moreover, the dynamic algorithms possess the property of practicability and are capable of working in real time, processing the incoming information during the course of the motion of the system and producing a result dynamically as the motion develops.

It is well known that, in the case of ill-posed linear problems, a classical Tikhonov n-th order regularization gives a high quality of approximation (reconstruction) in the case of a required smooth function. However, it does not allow of a qualitative reconstruction of non-differentiable functions which can contain kinks, close peaks, discontinuities and other singularities. Controls in dynamical systems can have singularities of just this kind. Stabilizing functionals, containing the norm of the Sobolev space W_p^n, have a strong stabilizing effect which inevitably leads to a smoothing out of the required function and to the loss of its fine structure. In particular, stabilizing functionals, containing the norm of the space L_p, $1 \leq p < \infty$, also lead to a fairly coarse approximation. The need therefore arises to construct stabilizers which are specially adapted to reconstruct non-smooth functions and functions with singularities. Up to the present time, several classes of stabilizing functionals, which have favourably proved themselves both for smooth and unsmooth reconstructed functions, have been proposed in variational regularization methods. In the case of functions of one variable, stabilizers containing a classical or generalized variation are often used in conjunction with some strictly convex norm such as, for example, the norm of the space L_p, $1 < p < \infty$.[14–20] Convergence in L_p, pointwise convergence, convergence of variations and, also, uniform convergence on segments of continuity of the required functions can be successfully achieved by this route. In the case of functions of several variables, stabilizers containing a generalized variation[21] and the norm of the space L_p, $1 \leq p < \infty$ are frequently used.[15,16,22–24] Here, convergence in L_p, pointwise convergence and convergence of the variations of the regularized approximations to the required function are successfully achieved. Stabilizers in the form of the norm of the Lipschitz space are used to obtain a uniform approximation of a continuous but, in general, non-differentiable function.[16] The use of the norm of the Sobolev space W_p^γ with fractional derivatives of the order $\gamma \in (0, 1)$ can turn out to be advisable both for reconstructing continuous as well as discontinuous required functions.[14,16]

It is shown below that, when stabilizers in the form of the sum of a classical variation and the norm of the space L_2 are used, it is possible to obtain pointwise convergence, convergence in L_2, convergence of variations and uniform convergence on the continuous segments of the required control which is reconstructed. In this sense, one may speak of the possibility of the numerical reconstruction of the fine structure of a required control.

1. Formulation of the problem

We will now describe the interesting aspect of the problem. Consider a controlled dynamical system, the behaviour of which, in a specified limited time interval $T = [t_0, \vartheta]$ $(-\infty < t_0 < \vartheta < +\infty)$, is described by the system of ordinary differential equations

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \in T, \quad x(t_0) = x_0 \tag{1.1}$$

where $R^n \ni x = x(t)$ is the vector of the phase state of the system at an instant $t \in T$, x_0 is the initial state of the system, $R^m \ni u = u(t)$ is the vector of the control acting on the system at an instant $t \in T$ (the control, parameter) and $f = f(t, x, u)$ is a given vector function which reflects the dynamical properties of the system. The permissible current values of the control action are subject to the given geometrical constraints

$$u(t) \in P \subset R^m, \quad t \in T \tag{1.2}$$

which characterize the possibilities of the control or reflect the known estimates for the permissible change of a parameter.

Suppose an observation during a time interval T is performed on a controlled dynamical system and its motion $x = x(t)$, the states of the system $x(t)$ are approximately measured at the corresponding current instants $t \in T$ and, at the same time, the results of these measurements $y(t)$ satisfy the following condition for the accuracy of the measurements

$$\|x(t) - y(t)\|_n \leq \delta, \quad t \in T \tag{1.3}$$

where $\|\cdot\|_m$ is the Euclidean norm in R^n and δ is a numerical parameter characterizing the accuracy of the measurements, $0 \leq \delta \leq \delta_0$.

The reconstruction problem consists of the approximate determination (reconstruction) of that realization $u = u(t)$ of the control action on the dynamical system which answers (corresponds) to the results of the observations using the results of the approximate measurements $y = y(t)$ of the observed motion of the system $x = x(t)$. At the same time, the result $u_\delta = u_\delta(t)$ of the reconstruction of the required control action $u = u(t)$ must be the more accurate, the smaller the errors in the measurements

$$\int \|u_\delta(t) - u(t)\|_m^2 dt \to 0, \quad \delta \to 0 \tag{1.4}$$

Unless otherwise stated, all the functions are considered when $t \in T$ and integration with respect to t is carried out over the interval T. The meaning of the concept of approximate reconstruction $u_\delta \approx u$ will then be varied and refined. Methods and reconstruction algorithms will be proposed which, besides the mean square approximation (1.4), ensure approximation in a certain stronger sense leading to the reconstruction of the fine structure of the required control. At the same time, it will be assumed that the *a priori* geometrical constraints P, imposed on the set of permissible controls, and the equations of the dynamics of the process (the function f) together with the initial state x_0 are known to the observer striving to solve the reconstruction problem.

We now consider the mathematical formulism of the problem. Suppose P is a convex compact set from R^m and U is the set of all measurable and square-integrable vector functions which, for almost all $t \in T$, belong to the compactum P

$$U = \{u \in E: u(t) \in P \text{ п.в. } t \in T\}, \quad E = L_2(T; R^m)$$

This set represents the set of all permissible controls in the problem considered.

To be specific, suppose the function f then has the form

$$f(t, x, u) = f_1(t, x) + f_2(t, x)u$$

The functions $f_1: T\times R^n \to R^n$ and $f_1: T\times R^n \to R^{n\times m}$ are continuous in the set $T\times R^n$ and, in this set, they satisfy the condition of sublinear growth and a local Lipschitz condition with respect to the variable x (see, for example, Refs 5 and 12). It is well known[5,12] that, in the case of these conditions, which the function f satisfies, a unique solution $x(\cdot)=x(\cdot;\, u)=x(t;\, u)$ of Cauchy problem (1.1) which is absolutely continuous in the interval T exists for each element $u\in U$. This solution will sometimes be called the motion of the dynamical system (1.1) generated by the control $u\in U$.

We now introduce the set of all possible motions of system (1.1) corresponding to all possible controls $u\in U$,

$$X = \{x(\cdot) = x(\cdot;\, u) : u\in U\}$$

For each motion $x(\cdot)\in X$, we introduce the set of all permissible controls corresponding to the given motion

$$U(x(\cdot)) = \{u\in U : x(\cdot) = x(\cdot;\, u)\}$$

and the set of all possible measurements of this motion

$$Y(x(\cdot), \delta) = \{y\in H : \|x(t) - y(t)\|_n \le \delta,\ t\in T\},\quad H = L_2(T;\, R^n)$$

The problem consists of constructing an algorithm which, using any permissible measurements of the current states of the observed motion of the system, approximately reconstructs a control which is in accord with the results of the observations of the motion. We identify the required algorithm with the family of mappings (methods)

$$D = \{D_\delta : 0\le\delta\le\delta_0\},\quad D_\delta : H\to E$$

The initial problem can now be formulated as follows: it is required to construct an algorithm $D=\{D_\delta: 0\le\delta\le\delta_0\}$ which, for any observed motion $x(\cdot)\in X$, possesses the regularizing property

$$r_\delta(x(\cdot))\to 0,\quad \delta\to 0$$

$$r_\delta(x(\cdot)) = \sup\{\rho[D_\delta(y), U(x(\cdot))] : y\in Y(x(\cdot), \delta)\}$$

$$\rho[D_\delta(y), U(x(\cdot))] = \min\{\|D_\delta(y) - \upsilon\|_E : \upsilon\in U(x(\cdot))\}$$

As will be shown below, in fact, $D_\delta(y)\to\hat{u}$ strongly in E, where $\hat{u}=\hat{u}(x(\cdot))$ is a certain element from the set $U(x(\cdot))$.

Before starting to solve the problem, we well note several algebraic and topological properties of the motions of the system and the sets which have been introduced into the treatment. The set U is convex, bounded and closed, and it is therefore weakly compact in the space E. The set X is compact in the space $C(T;\, R^n)$, and the weak convergence of the controls $u_k \rightharpoonup u_0$ in E implies the strong convergence of the motions $x(\cdot;\, u_k)\to x(\cdot;\, u_0)$ in $C(T;\, R^n)$. For each $x(\cdot)\in X$, the set $U(x(\cdot))$ is non-empty, convex, bounded, closed and therefore weakly compact in the space E and has a unique element of minimum E-norm.

In particular, it follows from what has been said above that the mapping

$$A : E\supset U\ni u\to x(\cdot;\, u)\in X\subset C(T;\, R^n)$$

is compact and, therefore, cannot have a continuous inverse mapping even if it is considered as a multivalued mapping. The ill-posed nature of the reconstruction problem and the need to invoke regularization methods to solve it follows from this.

We shall subsequently also consider the Banach space[14]

$$W = \{u\in E : V[u] < \infty\},\quad \|u\|_W = \|u\|_E + V[u]$$

where $V[u]$ is the total variation of the function $u: T\ni t\to u(t)\in R^m$:[25–27]

$$V[u] = \sup\left\{\sum_{i=1}^{l}\|u(t_i) - u(t_{i-1})\|_m\right\}$$

The supremum is chosen using all possible finite subdivisions of the segment T.

The space W is entirely continuously imbedded in E[14,27] (the imbedding operator is continuous, and it converts every bounded set from W into a pre-compact set from E). Every closed sphere in W is a closed set in E.[14] The pointwise limit of a bounded sequence of functions from W is also a function from W. The functional $V[\cdot]$ is lower semi-continuous with respect to pointwise convergence (if the sequence of functions of the bounded variation $\{v_k\}$ converges pointwise to a certain function of the bounded variation v_0, then $V[v_0]\le\liminf V[v_k]$). The norm $||\cdot||_W$ is lower semi-continuous with respect to pointwise convergence. The functional V[·] is lower semi-continuous with respect to convergence of bounded sequences from W in E (if $\{v_k\}$ is a bounded sequence from W which converges to a certain element $v_0\in E$ in E, then $v_0\in W$ and $V[v_0]\le\liminf V[v_k]$). The norm $||\cdot||_W$ is lower semi-continuous with respect to convergence of bounded sequences from W in E.

All the numerical quantities and spaces considered in this paper are assumed to be real, measurability and integrability are understood in the Lebesgue sense, the definitions of the functional spaces used are taken into account (for example, see Refs 12, 25 and 26), and proofs of the assertions formulated above are available (see, for example, Refs 14,25–29).

2. Solution of the reconstruction problem using the static method

We shall construct the required algorithm. For any $\delta \in [0, \delta_0]$, $y \in H$, we define the realization (value) of a method $D_\delta(y)$ according to the rule

$$D_\delta(y) = \upsilon \in \mathrm{D}\arg\min\{F_\alpha(y;\upsilon) : \upsilon \in U_W\} \tag{2.1}$$

$$F_\alpha^*(y) = \min\{F_\alpha(y;\upsilon) : \upsilon \in U_W\}, \quad U_W = U \cap W$$

$$F_\alpha(y;\upsilon) = \|x(\cdot;\upsilon) - y(\cdot)\|_H^2 + \alpha\Omega(u), \quad \Omega(u) = \|u\|_E^2 + V[u] \tag{2.2}$$

where α is a positive regularization parameter. It will be chosen depending on the magnitude δ of the error in the measurements.

Note several properties of the extremal problem (2.2) in advance. This problem is solvable for each fixed $\alpha > 0$, the set of its solutions (the set of minimizing elements) $U_\alpha^*(y)$ is non-empty and compact in E, and it is also weakly closed in E and closed in W. Every minimizing sequence in problem (2.2) converges strongly (weakly) in E to the set of solutions $U_\alpha^*(y)$, that is, every subsequence of a minimizing sequence which converges strongly (weakly) in E converges strongly (weakly) to a certain element of the set $U_\alpha^*(y)$. Proofs of analogous assertions are available (see Refs 27–29).

We shall call an element $\hat{u}$ of the set $Q \subseteq U$ which satisfies the condition

$$\Omega(\hat{u}) = \Omega_* = \inf\{\Omega(u) : u \in Q\}$$

a Ω-normal element of the set Q and we denote it by $\hat{u}(Q)$.

Theorem 1. *Suppose $U(x(\cdot)) \cap W \neq \emptyset$. Then, a unique Ω-normal element $\hat{u} = \hat{u}(U(x(\cdot)))$ exists in the set $U(x(\cdot))$. If the regularization parameter $\alpha = \alpha(\delta)$ satisfies the consistency conditions*

$$\delta^2\alpha(\delta)^{-1} \to 0, \quad \alpha(\delta) \to 0, \quad \delta \to 0 \tag{2.3}$$

then the algorithm D, consisting of methods (2.1), solves the reconstruction problem, that is, the convergence $r_\delta(x(\cdot)) \to 0$ holds for any observed motion $x(\cdot) \in X$ when $\delta \to 0$. Moreover, whatever realizations of the measurements $y_\delta \in Y(x(\cdot), \delta)$ occur here, the following convergences hold for the realizations of the algorithm $v_\delta = D_\delta(y_\delta)$ when $\delta \to 0$:1) $v_\delta \to \hat{u}$ strongly in E, 2) $v_\delta(t) \to \hat{u}(t)$ in R^m pointwise in T, 3) $V[v_\delta] \to V[\hat{u}]$, 4) $v_\delta(t) \to \hat{u}(t)$ in R^m uniformly with respect to t in any continuous segment of the function $\hat{u}$.

The main part of the theorem is proved using a well-known scheme.[17]

3. Solution of the reconstruction problem by the dynamic method

We shall initially describe the method for solving the problem in an informal manner. We shall follow a procedure for solving the problem which reduces it to a problem of the positional control of a suitable subsidiary control system-model. Conceptually, this procedure goes back to Refs 6,7.

A certain suitable control system-model

$$\dot{z}(t) = g(t, z(t), \upsilon(t)), \quad t \in T, \quad z(t_0) = z_0 \tag{3.1}$$

is constructed in advance, the current state of which at an instant $t \in T$ is described by a phase vector $z = z(t) \in R^n$; z_0 is the given initial state of the system-model, the vector of the control actions υ at the current instant $t \in T$ is restricted by the constraint $v(t) \in P$ and $g = g(t, z, v)$ is a given vector function which reflects the dynamical properties of the system-model.

A certain suitable control law is then constructed for the system-model (a positional strategy or a control procedure with a guide[5–8]) which will be identified with the function $V = V(t, \tau, y, z, v)$, that is defined for all possible values of the arguments $t \in T$, $\tau \in [t, \vartheta]$, $y \in R^n$, $z \in R^n$, $v \in R^m$ and takes values in the set $U[t, \tau]$, where $U[t, \tau]$ is the set of all measurable mappings $[t, \tau] \to P$.

Suppose some subdivision of the segment T is given by the points $t_i (t = 0, \ldots, 1)$, $t_0 < t_1 < \ldots < t_l$. According to this partitioning, a strategy Δ generates a motion $z_\Delta = z_\Delta(\cdot; t_0, z_0, \vartheta, V)$ of the system-model in the segment T

$$\dot{z}_\Delta(t) = g(t, z_\Delta(t), \upsilon_i(t)), \quad t_i \le t \le t_{i+1}, \quad i = 0, \ldots, l-1$$

$$z_\Delta(t_0) = z_0, \quad z_\Delta(t_i) = z_\Delta(t_i - 0), \quad i = 1, \ldots, l-1 \tag{3.2}$$

the control action $v_i = v_i(t)$, $t_i \le t \le t_{i+1}$ is generated (determined or calculated) at an instant t_i on the basis of the data $y(t_i)$ concerning the measurement of the state $x(t_i)$ of the observed system which has arrived up to this instant, the data $z_\Delta(t_i)$ concerning the state of the system-model and the data $v_{i-1}(t_i)$ concerning the control action of the model which has been realized in the preceding step, taken at the instant t_i, that is,

$$\upsilon_i = V(t_i, t_{i+1}, y(t_i), z_\Delta(t_i), \upsilon_{i-1}(t_i))$$

The method of obtaining the control action $v_{i-1}(t_i)$ when $i = 0$ is explained below.

It is found that, for a fairly wide range of problems, the system-model and the law controlling it can be chosen so that the realization of the control strategy

$$\upsilon_\delta^\Delta(\cdot) : \upsilon_\delta^\Delta(t) = \upsilon_i(t), \quad t_i \le t \le t_{i+1}, \quad i = 0, \ldots, l-1$$

will be close in a certain sense to the required control of system (1.1) only if the error in the measurements δ and the diameter $d(\Delta)$ of the partitioning Δ are sufficiently small and they are matched in a certain manner. The magnitude of the diameter $d(\Delta)$ of the partitioning Δ is assumed to depend on δ.

Note that the variable of the state of the subsidiary system-model and the control action variable of this system can be considered as the internal variables of the required algorithm D which solves the reconstruction problem by the dynamic method. These internal variables can be independently calculated using a computer. A copy of the initial system can frequently be taken as the system-model but other methods of selecting the system-model are possible, and this question is solved separately in each specific case.

It will henceforth be assumed that $g(t, z, v) = f(t, z, v)$. From a practical computational point of view, it is convenient to solve Cauchy problems (3.1) or (3.2) using a discrete Euler scheme.[5–8] The strategy V is constructed from these considerations such that the motion of the system-model (3.1), appearing from the state z_0 at the instant t_0 under the action of this strategy, tracks the dynamics of the measurements $y(t_i)$ of the observed states $x(t_i)$ of system (1.1) in a particular sense. The idea of constructing such a law for controlling system-model (3.1) is incorporated in the well known extremal displacement method from positional control theory,[5–8] which is locally regularized by one of the well-known methods of regularization.[1–3] The Tikhonov regularization method will be used in this paper.

We will now formally define the required algorithm D and identify it with the family of methods D_δ^Δ:

$$D = \{D_\delta^\Delta : \Delta \in \Sigma,\ 0 \le \delta \le \delta_0\}$$

where Σ is the set of all finite partitionings Δ of the segment T. Each method D_δ^Δ constitutes a set of mappings $D_{\delta i}^\Delta$:

$$D_\delta^\Delta = \{D_{\delta i}^\Delta : i = 0, \dots, l-1\}$$

$$D_{\delta i}^\Delta : R^n \times R^n \times R^m \to U[t_i, t_{i+1}] \cap W[t_i, t_{i+1}]$$

where $W[t_i, t_{i+1}]$ is the set of all functions $[t_i, t_{i+1}] \to R^m$ of a bounded variation in $[t_i, t_{i+1}]$. We call the function $v_\delta^\Delta : T \to R^m$, which is defined by the equalities

$$v_\delta^\Delta(t) = v_i(t) = D_{\delta i}^\Delta(y(t_i), z(t_i), v_{i-1}(t_i)), \quad t_i \le t < t_{i+1}, \quad i = 0, \dots, l-1$$

$$v_\delta^\Delta(\vartheta) = v_{l-1}(\vartheta)$$

the realization of the method D_δ^Δ for a measurement $y \in Y(x(\cdot), \delta)$ and we denote it by the symbol $D_\delta^\Delta(y)$. The value $z = z(t_i)$ of the internal variable z of the method D_δ^Δ at the instant t_i is uniquely formed on the basis of the available information $t(t_i)$ concerning the motion of system (1.1) which has been accumulated up to this instant and the control actions $v_j (j = 0, \dots, i-1)$ of this method which have been realized. We shall formulate the rule for forming the variable z and the control action $v_{-1}(t_0)$ of the method D_δ^Δ below, where the specific technique for constructing the methods of the algorithm will be discussed.

The initial reconstruction problem can now be formulated as follows: it is required to construct an algorithm

$$D = \{D_\delta^\Delta : \Delta \in \Sigma,\ 0 \le \delta \le \delta_0\}$$

which, for particular matchings between the quantities $d(\Delta)$ and δ, possesses the regularizing property $r_\delta^\Delta(x(\cdot)) \to 0$ when $\delta \to 0$ for any observed motion $x(\cdot) \in X$, where

$$r_\delta^\Delta(x(\cdot)) = \sup\{\rho[D_\delta^\Delta(y), U(x(\cdot))] : y \in Y(x(\cdot), \delta)\}$$

Before starting to construct the specific algorithm that solves this problem, we will assume that a procedure is available to the observer which enables him to determine the value $u_0 = u(t_0)$ of the real control $u(\cdot) = u(\cdot; x(\cdot))$ from the set $U(x(\cdot))$ that generates the observed motion $x(\cdot) \in X$. We will denote this approximate value by the symbol $u_h^0 = u_h^0(x(\cdot))$, where h is the accuracy of the approximation obtained $\|u_h^0 - u_0\|_m \le h$. The magnitude of the parameter h will be subject to the value of δ and the value of the parameter $h \in [0, h_0]$ will be chosen as a function of the value, δ, of the accuracy of the measurements of the states of the system, $h = h(\delta)$. This informal condition concerning the possibility of finding an approximation u_h^0 will be formally used in algorithm D for assigning the vector $v_{-1}(t_0) = u_h^0$.

We now introduce some notation

$$U[t, \tau; w] = \{u \in U[t, \tau] : u(t) = w\}$$

$$H_{\delta i}^\Delta(v; y, z) = 2\left\langle z - y, \int_{t_i}^{t_{i+1}} f_2(\tau, y) v(\tau) d\tau \right\rangle + \alpha(\delta)\Omega_{t_i}^{t_{i+1}}(v)$$

$$\hat{H}_{\delta i}^\Delta(y, z, w) = \min\{H_{\delta i}^\Delta(v; y, z) : v \in U[t_i, t_{i+1}; w] \cap W[t_i, t_{i+1}]\}$$

$$\Omega_t^\tau(v) = \int_t^\tau \|v(\eta)\|_m^2 d\eta + V_t^\tau[v]$$

where $V_t^\tau[v]$ is the total variation of the function $[t,\tau] \to R^m$ in $[t,\tau]$, when $t=\tau$ by definition we put $V_t^t[v]=0$ and $\langle\cdot,\cdot\rangle$ is a scalar product in R^n.

We will now construct the specific algorithm. For any

$$\delta \in [0, \delta_0], \quad \Delta \in \Sigma, \quad i \in \{0, \dots, l-1\}, \quad y \in R^n, \quad z \in R^n, \quad w \in P$$

we define the value of the mapping $D_{\delta i}^\Delta$ at a point (y, z, w) according to the rule $D_{\delta i}^\Delta(y, z, w) = v_\delta^i$, where v_δ^i is an element of the set $U[t_i, t_{i+1}; w] \cap W[t_i, t_{i+1}]$ which satisfies the condition

$$\hat{H}_{\delta i}^\Delta(y, z, w) \le H_{\delta i}^\Delta(v_\delta^i; y, z) \le \hat{H}_{\delta i}^\Delta(y, z, w) + \varepsilon(\delta) \tag{3.3}$$

and $\alpha=\alpha(\delta)$ and $\varepsilon=\varepsilon(\delta)$ are the positive regularization parameters of the algorithm.

We define the value $z(t)$ of the internal variable z of the algorithm at an instant $t \in T$ as follows: if $t=t_0$, then we put $z(t_0)=y(t_0)$ and, if $t \in [t_i, t_{i+1})$ for any $i \in \{0, \dots, l-1\}$, then we put

$$z(t) = z(t_i) + \int_{t_i}^{t} f(\tau, y(t_i), D_{\delta i}^\Delta(y(t_i), z(t_i), v_{i-1}(t_i)))(\tau)d\tau \tag{3.4}$$

Note that, when $i>1$, there is no need to calculate the values of $z(t_j)$, $v_j - 1(t_j)$ $(j = 0, \dots, i)$ each time and it is sufficient to recall the last values of $z(t_i)$ and $v_{i-1}(t_i)$ when solving the problem. We also note that the extremal problem (3.3) is easier than problem (2.2) from a computational point of view.

We will now describe the step-by-step operation of the algorithm in time. Suppose some motion $x(\cdot) \in X$ is observed. Prior to the commencement of the reconstruction process, the functions $\alpha=\alpha(\delta)$, $\varepsilon=\varepsilon(\delta)$, $h=h(\delta)$ and the partititioning Δ of the segment T with a diameter $d(\Delta) \le \varphi(\delta)$ are fixed and the error level δ, at which the states of the observed motion will be measured, becomes known.

3.1. The step i = 0

At the instant t_0, information in the form of the measurement $y(t_0)$ of the state $x(t_0)$ of the observed motion $x(\cdot) \in X$ of the system and an approximate value $u_h^0 = u_h^0(x(\cdot))$ of the real control which generates this observed motion is received by the observer. Putting

$$y = y(t_0), \quad z = z(t_0), \quad w = u_h^0$$

the observer, at the instant t_0 and using rule (3.3), finds the part $v_\delta^0 = D_{\delta 0}^\Delta(y, z, w)$ of the realization $v_\delta^\Delta = D_\delta^\Delta(y)$ of the method D_δ^Δ which is taken as an approximation to the required control in the time interval $t_0 \le t \le t_1$. The value $v_\delta^\Delta(t_1)$ of the control which has been found is stored in order to carry out the next step. Then, using rule (3.4), the state $z(t_1)$ of the system-model (an internal variable of the algorithm) is determined and stored in order to carry out the next step.

3.2. The step i = 1

At the instant t_1, information in the form of the measurement $y(t_1)$ of the state $x(t_1)$ of the observed motion $x(\cdot)$ of the system is received by the observer. Putting

$$y = y(t_1), \quad z = z(t_1), \quad w = v_\delta^0(t_1)$$

at the instant t_1 and using rule (3.3), the observer finds the part $v_\delta^1 = D_{\delta 1}^\Delta(y, z, w)$ of the realization v_δ^Δ of the method D_δ^Δ which is taken as an approximation of the required control in the time interval $t_1 \le t \le t_2$. The value $v_\delta^1(t_2)$ of the control which has been found is stored in order to carry out the next step. Then, using rule (3.4), the state $z(t_2)$ of the system-model (an internal variable of the algorithm) is determined and stored in order to carry out the next step.

The following steps $i=2, \dots, l-1$ are similar to the step $i=1$. The total realization of the method $v_\delta^\Delta = D_\delta^\Delta(y)$ will therefore be obtained in a stepwise manner during the course of the process (in the dynamics) up to the final instant $t_l = \vartheta$. From the description of the operation of the algorithm in time, it is clear that it can be performed in real time.

Theorem 2. *Suppose*

$$U(x(\cdot)) \cap U[t_0, \vartheta; u_0] \cap W \ne \emptyset$$

Then, a unique Ω-normal element $\hat{u}$ exists in the set $U(x(\cdot)) \cap U[t_0, \vartheta; u_0]$. If the regularization parameters $\alpha=\alpha(\delta)$, $\varepsilon=\varepsilon(\delta)$, $h=h(\delta)$ and the estimate $\varphi(\delta)$ of the diameter $d(\Delta)$ of the partitioning Δ of the segment T satisfy the conditions

$$(\varphi(\delta) + \varepsilon(\delta) + \delta)\alpha(\delta)^{-1} \to 0, \quad \alpha(\delta) \to 0, \quad \varepsilon(\delta) \to 0, \quad \varphi(\delta) \to 0, \quad h(\delta) \to 0$$

when $\delta \to 0$, then the algorithm D, consisting of methods (3.3) and (3.4), *solves the reconstruction problem, that is, the convergence $r_\delta^\Delta(x(\cdot)) \to 0$ holds for any observed motion $x(\cdot) \in X$ when $\delta \to 0$. Moreover, in when realizing the methods $v_\delta^\Delta = D_\delta^\Delta(y)$ and whatever measurements $y \in Y(x(\cdot), \delta)$ have taken place here, the following convergences hold when $\delta \to 0$: $v_\delta^\Delta \to \hat{u}$ strongly in E, $v_\delta^\Delta(t) \to \hat{u}(t)$ in R^m pointwise in T, $V[v_\delta^\Delta] \to V[\hat{u}]$ and $v_\delta^\Delta(t) \to \hat{u}(t)$ in R^m uniformly with respect to t from any continuous segment of the fuction $\hat{u}$.*

Proof. We will now prove the first part of the theorem. Suppose

$$Q = U(x(\cdot)) \cap U[t_0, \vartheta; u_0] \cap W \neq \emptyset \quad \text{и} \quad u_* \in Q$$

Then, $\Omega(u_*) < \infty$ and all Ω-normal solutions, if they exist, must be contained in the set

$$Q^* = \{u \in Q : \Omega(u) \leq \Omega(u_*)\} \neq \emptyset$$

Suppose $\{u_k\} \subset Q^*$ is an arbitrary sequence which minimizes the functional Ω on the set Q^*, that is,

$$\Omega(u_k) \to \Omega_* = \inf\{\Omega(u) : u \in Q^*\}$$

On the basis of Helly's first and second theorems and, also, Lebesgue's theorem on passing to the limit under an integral sign, we can assume, without loss of generality and passing to a subsequence when necessary, that an element $u^* \in Q^*$ exist such that $u_k \to u^*$ in E and $u_k(t) \to u^*(t)$ in R^m pointwise in T. Taking account of the lower semicontinuity of the functional Ω with respect to pointwise convergence in T, we obtain

$$\Omega_* \leq \Omega(u^*) \leq \liminf \Omega(u_k) = \lim \Omega(u_k) = \Omega_*$$

Hence, $\Omega(u^*) = \Omega_*$ and the non-emptiness of the set of Ω-normal elements in the set Q^* is proved. Next, since the sets $U(x(\cdot))$, $U[t_0, \vartheta; u_0]$, W are convex, the intersection Q of these sets is also convex. The strict convexity of the functional Ω in Q follows from the strict convexity of the norm in Hilbert space E and the convexity of the total variation in W. The uniqueness of the element minimizing the functional Ω in Q, that is, the uniqueness of the Ω-normal element, follows from the strict convexity of Ω in Q. The first part of the theorem is proved.

We shall henceforth assume that the set Q is non-empty for any $x(\cdot) \in X$ and, consequently, a unique Ω-normal element $\hat{u} = \hat{u}(x(\cdot))$ will always exist in the set Q.

We will now prove the following part of the theorem. We fix an arbitrary motion $x(\cdot) \in X$ and some function $\alpha = \alpha(\delta)$, $\varepsilon = \varepsilon(\delta)$, $h = h(\delta)$, $\varphi = \varphi(\delta)$ which satisfy the correspondences from the condition of the theorem. To prove the remaining part of the theorem, it is sufficient to show that, whatever the numerical sequence $\{\delta_k\} \subset [0, \delta_0]$, $\delta_k \to 0$, the sequence of elements $\{y_k\}$, $y_k \in Y(x(\cdot), \delta_k) k \in N$ and the sequence of partitionings $\{\Delta_k\} \subset \Sigma$, $d(\Delta_k) \leq \varphi(\delta_k)$, $k \in N$, the following convergences hold: $\upsilon_k = \upsilon_{\delta_k}^{\Delta_k} = D_{\delta_k}^{\Delta_k}(y_k) \to \hat{u} = \hat{u}(x(\cdot))$ in E, $\upsilon_k(t) \to \hat{u}(t)$ in R^m pointwise in T, and $V[\upsilon_k] \to V[\hat{u}]$.

We now fix some sequences $\{\phi_k\}$, $\{y_k\}$, $\{\Delta_k\}$ which satisfy the above mentioned conditions and show that the above mentioned convergences hold. Taking account of the rule for the formation of a realization of the algorithm $\upsilon_k = D_{\delta_k}^{\Delta_k}(y_k)$, the following estimate can be obtained for the functional $\Lambda_k = \Lambda_k(t; x, z_k, \alpha, \upsilon_k, \hat{u})$

$$\Lambda_k(t_i^{(k)}) \leq \nu_k, \quad i = 0, \ldots, m^{(k)}$$

$$\Lambda_k(t) = \|x(t) - z_k(t)\|_n^2 + \alpha(\delta_k)\Omega_{t_0}^t(\upsilon_k) - \alpha(\delta_k)\Omega_{t_0}^t(\hat{u})$$

$$\nu_k = \delta_k^2 + \kappa_k(\vartheta - t_0), \quad \kappa_k = C_0(\varphi(\delta_k) + \varepsilon(\delta_k) + \delta_k) \tag{3.5}$$

where C_0 is a certain positive constant, which is independent of $k \in N$ and is solely determined by data on the system and the problem which is known *a priori*, and z_k is the motion of the system-model, generated by the realization of the algorithm υ_k (the construction of this motion is described in (3.4)).

The inequalities

$$\max\{\|x(t) - z_k(t)\|_n^2 : t \in T\} \leq \nu_k^* \tag{3.6}$$

$$\Omega(\upsilon_k) \leq \Omega(\hat{u}) + \nu_k \alpha(\delta_k)^{-1} \tag{3.7}$$

follow from the estimate (3.5), where

$$\nu_k^* = C_1(\nu_k + \alpha(\delta_k) + \varphi(\delta_k)^2)$$

and C_1 is a certain positive constant which is analogous to the constant C_0. Since $\nu_k^* \to 0$, we obtain the convergence

$$z_k(t) \to x(t) \ \text{в} \ R^n \text{ uniformly with respect to } t \in T \tag{3.8}$$

from inequality (3.6).

According to the condition of the theorem $\nu_k \alpha(\delta_k)^{-1} \to 0$. The sequence $\nu_k \alpha(\delta_k)^{-1}$ is therefore bounded and a number $C_3 \geq 0$ exists such that $\nu_k \alpha(\delta_k)^{-1} \leq C_3$ for any $k \in N$. From inequality (3.7) we then obtain

$$\Omega(\upsilon_k) \leq \Omega(\hat{u}) + C_3, \quad \forall k \in N \tag{3.9}$$

The boundedness of the sequence $\{\upsilon_k\}$ in E and W follows from inequality (3.9).

Then, on the basis of Helly's first and second theorems and Lebesgue's theorem concerning passing to the limit under an integral sign, the compactness of the imbedding of W in E, the weak compactness of the set U in E and the lower semicontinuity of the total variation with respect to pointwise convergence, we can assume, without loss of generality, that the convergences

$$\upsilon_k \to \upsilon_* \ \text{в} \ E \tag{3.10}$$

$$\upsilon_k(t) \to \upsilon_*(t) \ \text{в} \ R^m \text{ point wise in } T \tag{3.11}$$

$$V[v_*] \le \liminf V[v_k] \tag{3.12}$$

hold for a certain element $v_* \in U \cap W$.

Taking account of equality (3.4), the functions z_k can be represented in the form

$$z_k(t) = z_k(t_0) + \int_{t_0}^{t} f(\tau, \bar{y}_k(\tau), v_k(\tau))d\tau, \quad \forall t \in T, \quad \forall k \in N \tag{3.13}$$

where $\bar{y}_k$ is a piecewise-constant completion of the mesh function $y_k(t_1)(i = 0, \ldots, l)$ in the segment T, that is,

$$\bar{y}_k(t) = y_k(t_i), \quad t_i \le t < t_{i+1}, \quad i = 0, \ldots, l-1, \quad \bar{y}_k(\vartheta) = y_k(\vartheta)$$

It follows from the estimate

$$\|\bar{y}_k(t) - x(t)\|_n \le \delta_k + C_4\varphi(\delta_k)$$

that

$$\bar{y}_k(t) \to x(t) \text{ в } R^n \text{ uniformly in } T \tag{3.14}$$

From the form of the function

$$f(\tau, \bar{y}_k(\tau), v_k(\tau)) = f_1(\tau, \bar{y}_k(\tau)) + f_2(\tau, \bar{y}_k(\tau))v_k(\tau)$$

the convergence (3.14) and the weak convergence $v_k \rightharpoonup v_*$ in E, which follows from the convergence (3.10) or (3.11), we obtain the convergence

$$\int_{t_0}^{t} f(\tau, \bar{y}_k(\tau), v_k(\tau))d\tau \to \int_{t_0}^{t} f(\tau, x(\tau), v_*(\tau))d\tau, \quad \forall t \in T \tag{3.15}$$

We now take the limit in equality (3.13) on the basis of relations (3.8) and (3.15). After taking to the limit, we obtain the equality

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau), v_*(\tau))d\tau, \quad \forall t \in T$$

This equality means that

$$x(\cdot) = x(\cdot; v_*), \quad v_* \in U(x(\cdot)) \cap W$$

From the convergence (3.11), we have $v_k(t_0) = u^0_{h(\delta_k)} \to u_0$ in R^m and, therefore,

$$v_*(t_0) = u_0, \quad v_* \in U(x(\cdot)) \cap U[t_0, \vartheta; u_0] \cap W$$

Since $\hat{u}$ is a Ω-normal element in the set $U(x(\cdot)) \cap U[t_0, \vartheta; u_0]$, $\Omega(\hat{u}) \le \Omega(v_*)$. Then, taking account of relations (3.7), (3.10) and (3.12), we have

$$\Omega(\hat{u}) \le \Omega(v_*) \le \liminf \Omega(v_k) \le \limsup \Omega(v_k) \le \Omega(\hat{u})$$

From the resulting chain of inequalities we obtain

$$\Omega(v_k) \to \Omega(v_*) = \Omega(\hat{u})$$

By virtue of the uniqueness of the Ω-normal element in the set $U(x(\cdot)) \cap U[t_0, \vartheta; u_0]$, we obtain the matching of the elements $v_* = \hat{u}$ as elements of the space W (and this means also the pointwise matching). Moreover, the convergence $V[u_k] \to V[\hat{u}]$ follows from the convergence (3.10) and the convergence $\Omega(v_k) \to \Omega(\hat{u})$. Hence, the expected convergences are obtained: $v_k \to \hat{u}$ in E, $v_k(t) \to \hat{u}(t)$ in R^m pointwise in T and $V[u_k] \to V[\hat{u}]$. Since the sequences $\{\delta_k\}$, $\{y_k\}$, $\{\Delta_k\}$ were chosen arbitrarily, the convergences

$$v^{\Delta}_{\delta} \to \hat{u} \text{ in } E, \quad v^{\Delta}_{\delta}(t) \to \hat{u}(t) \text{ in } R^m \text{ pointwise in } T, \; V[v^{\Delta}_{\delta}] \to V[\hat{u}] \tag{3.16}$$

hold when $\delta \to 0$.

When there is a strong convergence $v^{\Delta}_{\delta} \to \hat{u}$ in E, it is easily shown by contradiction that $r^{\Delta}_{\delta}(x(\cdot)) \to 0$ when $\delta \to 0$.

It follows from the convergence (3.16) and well-known results[14,19,20] that $v^{\Delta}_{\delta}(t) \to \hat{u}(t)$ in R^m uniformly with respect to t from any segment in which the function $\hat{u}$ is continuous. This completes the proof.

4. Numerical modelling

We will now present the results of numerical modelling using the dynamic reconstruction of a control in the system

$$\dot{x}(t) = u(t)\sin x(t), \quad t \in T = [t_0, \vartheta], \quad x(t_0) = x_0, \quad x \in R \tag{4.1}$$

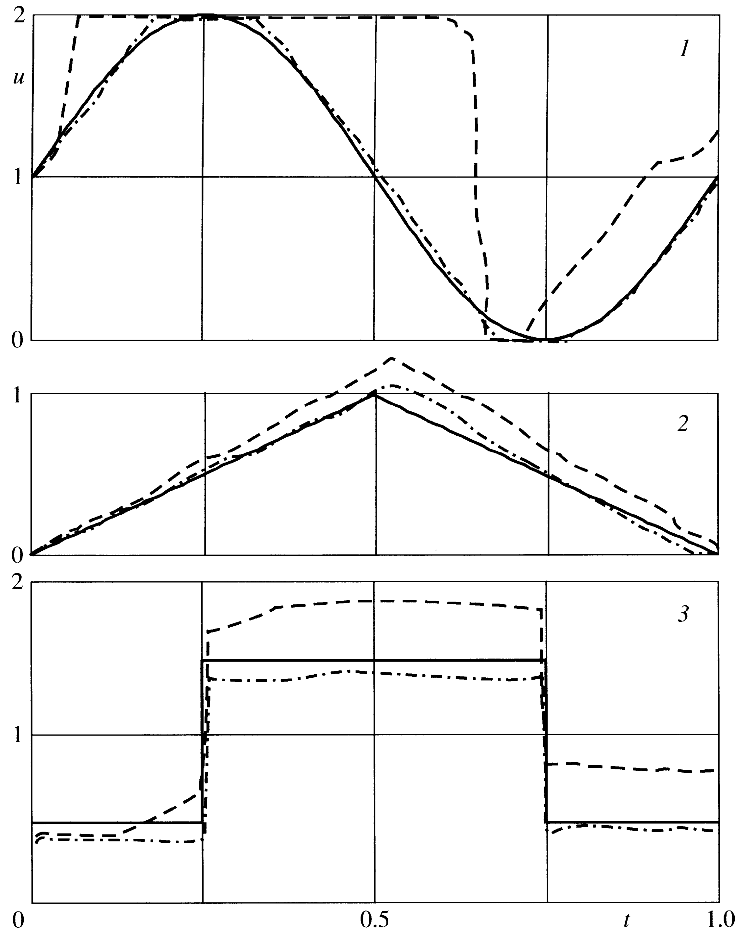

Fig. 1.

The segment $[\mu_1, \mu_2] \subset R$ represents the set of geometric constraints on the control P and the approximate measurement of the states of the system was modeled by the relation

$$y(t) = x(t) + \delta \sin(\omega t), \quad t \in T, \quad \omega = \text{const}$$

Calculations were carried out for the following parameters of the problem

$$t_0 = 0, \quad \vartheta = 1, \quad x_0 = 1, \quad \mu_1 = 0, \quad \mu_2 = 2, \quad \omega = 1$$

The model controls reconstructed were the following three functions:

1) $u = u_{(1)}(t) = 1 + \sin 2\pi t$ (a smooth control);
2) $u = u_{(2)}(t) = 1 - |2x - 1|$ (a continuously piecewise smooth control);
3) $u = u_{(3)}(t) = \begin{cases} 0.5, & \text{if } t \in [0, 1/4] \quad \text{are} \quad t \in [3/4, 1], \\ 1.5, & \text{if } t \in [1/4, 3/4] \, (\text{discontinuous control}) \end{cases}$

The extremal controls in the intervals of the partitioning were found by the method of projection of a subgradient[27–29] with a number of iterations I.

The reconstructed control is shown by the solid curve in the Fig. 1. The control *1* was reconstructed for the following parameter values

$$\alpha = 0.0001, \quad d(\Delta) = 0.004, \quad I = 9000$$

The reconstruction when $\delta = 0.46$ is shown by the dashed line and the reconstruction when $\delta = 0.02$ by the dot-dash line. The reconstruction of control *2* was carried out for

$$\alpha = 0.001, \quad d(\Delta) = 0.004, \quad I = 7700$$

The reconstruction when $\delta = 0.08$ is shown by the dashed line and the reconstruction when $\delta = 0.005$ by the dot-dash line. The reconstruction of control *3* was carried out for

$$\alpha = 0.0007, \quad d(\Delta) = 0.004, \quad I = 9000$$

The dashed line was obtained when $\delta = 0.46$ and the dot-dash line when $\delta = 0.01$.

The modelling shows that the fine structure of a control can be reconstructed to a considerable extent.

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